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Some observations on skin friction and velocity profiles in fully developed pipe and channel flows

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Measurements of skin friction and mean-velocity profiles have been made in fully developed flows in pipes and channels in the Reynolds number range $1000 < Re < 10000$. These measurements, and observations of hot-wire signals, indicate rather remarkable differences between two-dimensional and axially symmetric flows and also make it difficult to give a precise definition of the term 'fully developed turbulent flow'.

1. Introduction

The experiments described in this paper were undertaken in connexion with a study of reverse transition in turbulent boundary layers (Patel & Head 1968). The results, however, appear to be of sufficient general interest to warrant separate publication. Measurements of skin friction and mean-velocity profiles have been made in fully developed pipe and channel flows over a range of Reynolds number that includes the transition from fully laminar to fully turbulent flow. The results indicate significant, and (so far as the authors are aware) hitherto unsuspected, differences between two-dimensional flow in a parallel channel and axially symmetric flow in a circular pipe.

In the past, three separate criteria have been used to characterize fully developed turbulent flow in pipes and channels. Fully turbulent flow has been said to exist: (*a*) when the skin-friction coefficient is related to the Reynolds number by an established law known to hold at high Reynolds numbers; (*b*) when the velocity distribution in the wall region follows the well known law of the wall; or (*c*) when the flow is continuously turbulent, i.e. no intermittency is present. Results presented in this paper suggest that, as the Reynolds number is increased from some value corresponding to laminar flow, the three criteria first become applicable at different Reynolds numbers. Furthermore, the results obtained from pipe flow are quite different from those obtained from channel flow. There is therefore some uncertainty concerning not only the value of 3000 often quoted as the minimum Reynolds number for which fully turbulent flow can exist in a pipe, and a similar value for a channel, but also the precise meaning to be attached to the term 'fully developed turbulent flow'.

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2. Mean-flow measurements in pipes

Two pipes, $\frac{1}{4}$ in. and $\frac{1}{2}$ in. in internal diameter, shown schematically in figure 1, were used in these experiments. Air was supplied from a standard 100 psi laboratory compressed-air plant *via* a control valve. The inlet lengths upstream of the measuring station for the two pipes (218 diameters for the $\frac{1}{4}$ in. pipe and 228 diameters for the $\frac{1}{2}$ in. pipe) were sufficient for the flow to be fully developed over the test lengths. The pressure differences required to compute skin friction were measured between stations 1 and 2 indicated in figure 1. The rate of flow

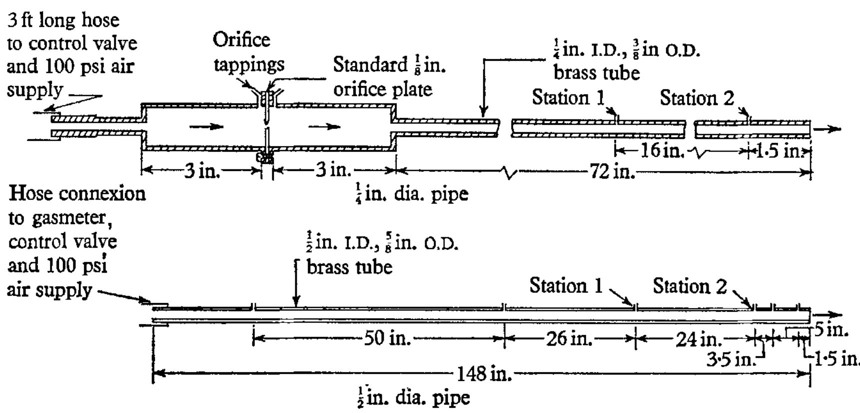

FIGURE 1. Pipe arrangements.

required to calculate the Reynolds number was measured by means of a standard $\frac{1}{8}$ in. diameter orifice for the $\frac{1}{4}$ in. pipe and by means of a Parkinson Cowan gasmeter for the $\frac{1}{2}$ in. pipe. The discharge coefficients of the orifice used in the former case were obtained from the curves given by Ower & Pankhurst (1966). Some additional measurements were also made in a 2 in. internal diameter pipe. This pipe has already been described in detail by Preston (1954) and by Patel (1965). Here the air was supplied by a centrifugal blower and volume flow was measured by means of $1\frac{1}{4}$ in. diameter standard orifice.

The variation of skin-friction coefficient with Reynolds number, based on the pipe diameter D and average velocity $\overline{U}$, is shown in figure 2. It will be seen that for Reynolds numbers less than approximately 2000 the measurements agree with the laminar flow result

$$C_f = 16/Re, \tag{1}$$

and for Reynolds numbers greater than about 3000 the measurements are in agreement with the well known Blasius friction law

$$C_f = 0{\cdot}079 Re^{-\frac{1}{4}}, \tag{2}$$

which is generally accepted as being applicable up to a Reynolds number of 10^5. The value of 2000 for the lower critical Reynolds number is in good agreement with that suggested by Gilbrech & Hale (1965) and others.

Mean-velocity distributions were measured $\frac{1}{8}$ in. upstream of exit in the $\frac{1}{4}$ in. and $\frac{1}{2}$ in. pipes at several Reynolds numbers within and on either side of the range characterizing transition from laminar to turbulent flow. The profiles were obtained by means of a 0·042 in. diameter circular pitot and the measurements were corrected for displacement effects using the results of MacMillan (1957). The results are shown in figure 3 on a linear scale and in figure 4 in the

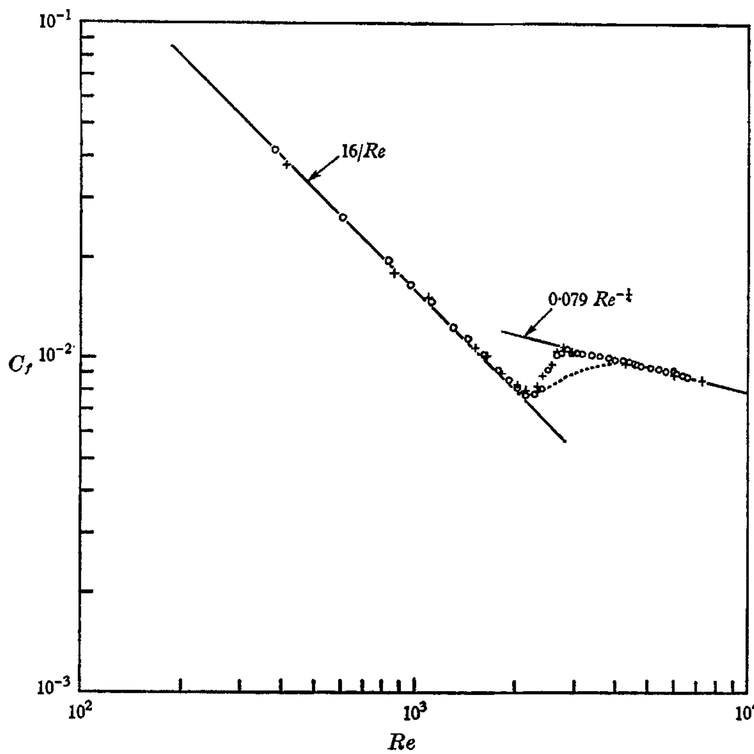

FIGURE 2. Skin friction in pipe flow. ○, $\frac{1}{4}$ in. diameter pipe; +, $\frac{1}{2}$ in. diameter pipe; - - - -, Nikuradse.

semi-logarithmic representation using the friction velocity $U_\tau = \surd(\tau_w/\rho)$ and ν/U_τ as scales of velocity and length. The significance of the latter figure will be discussed in a subsequent section, but from figure 3 it will be seen that at Reynolds numbers less than the lower critical value of about 2000, the measured profiles agree with the parabolic profile representing the exact laminar solution. A check on the overall accuracy of the measurements was provided by the fact that the average velocities computed from the measured velocity profiles were in excellent agreement with those measured by the orifice in the $\frac{1}{4}$ in. pipe and by the gasmeter in the $\frac{1}{2}$ in. pipe.

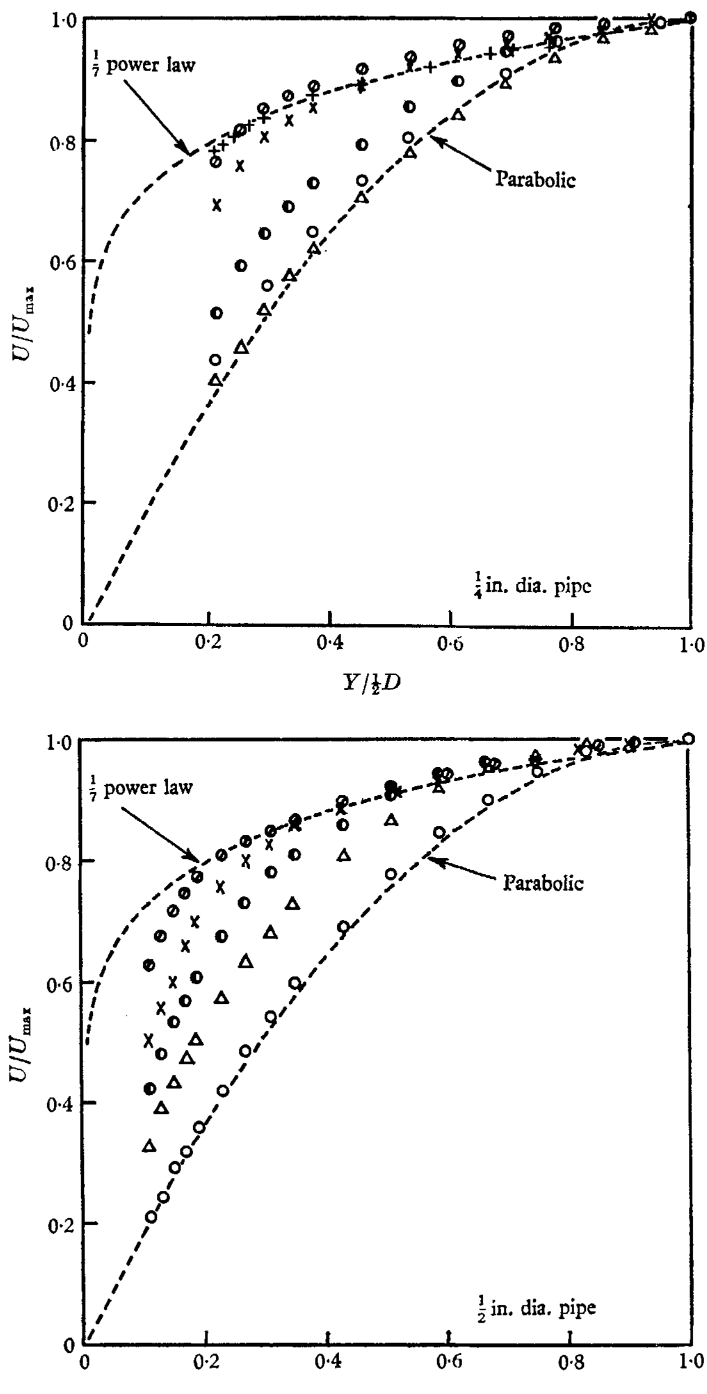

FIGURE 3. Velocity profiles in pipes. Values of Re in the $\frac{1}{4}$ in. pipe: ○, 1400; △, 2015; ◐, 2440; ×, 2680; ⊘, 3070; +, 4060. Values of Re in the $\frac{1}{2}$ in. pipe: ○, 1740; △, 2440; ◐, 2615; ×, 2975; ⊘, 4430.

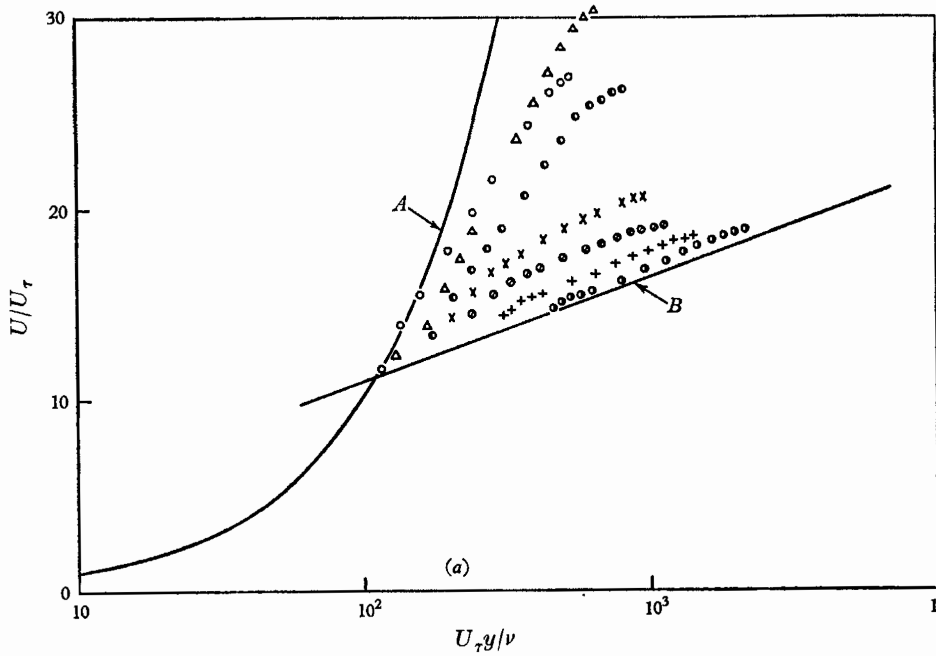

FIGURE 4(*a*). Velocity profiles in the $\frac{1}{4}$ in. diameter pipe. Values of *Re*: ○, 1400; △, 2015; ◐, 2440; ×, 2680; ⊘, 3070; +, 4060; ◑, 6300. Curve *A*, $U/U_\tau = U_\tau y/\nu$; curve *B*, $U/U_\tau = 5{\cdot}5 \log_{10} U_\tau y/\nu + 5{\cdot}45$.

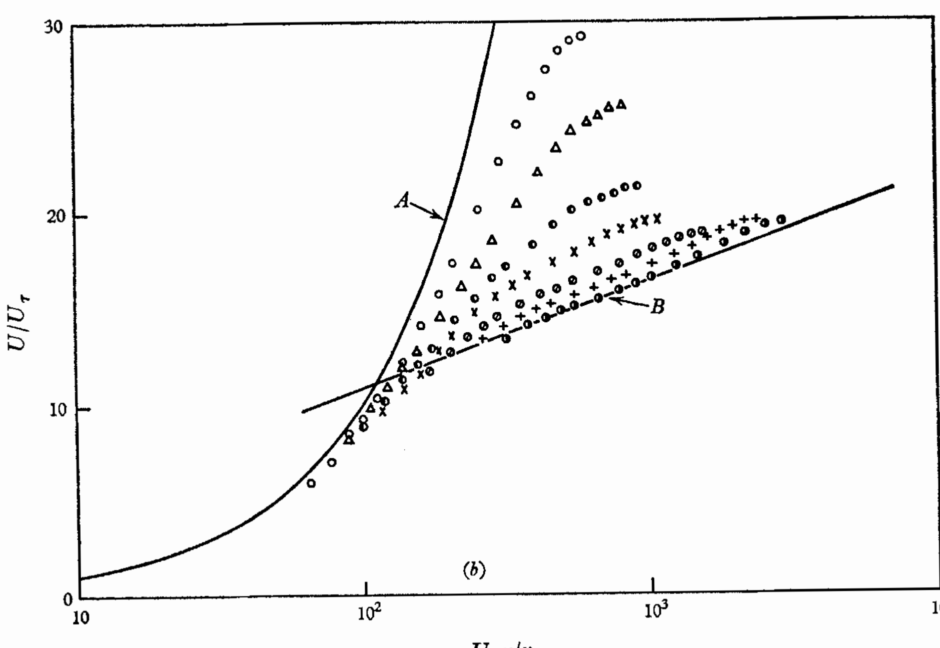

FIGURE 4(*b*). Velocity profiles in the $\frac{1}{2}$ in. diameter pipe. Values of *Re*: ○, 1740; △, 2440; ◐, 2615; ×, 2975; ⊘, 4430; +, 7260; ◑, 9200. Curve *A*, $U/U_\tau = U_\tau y/\nu$; curve *B*, $U/U_\tau = 5{\cdot}5 \log_{10} U_\tau y/\nu + 5{\cdot}45$.

3. Mean-flow measurements in a channel

Experiments were carried out in a rectangular parallel-sided channel $\frac{1}{4}$ in. high and 12 in. wide, the details of which are shown in figure 5. The aspect ratio of 48 was considered large enough for the flow to be assumed two-dimensional. In fact, detailed span-wise measurements showed that the effective width of the channel was only $2\frac{1}{2}\%$ less than the actual width. The overall length of the channel was 72 in., i.e. 288 channel heights, and the first measuring station was located 203·5 channel heights downstream of entry, so that the flow had a sufficient upstream length to become fully developed before reaching the test section. The pressure drop was measured over a distance of 16 in. between stations 1 and 2 shown in figure 5. The volume flow rate was measured by means of a gasmeter at low rates of flow and by means of an orifice plate at high rates of flow. Some additional measurements were also made in a $\frac{1}{4}$ in. channel twice as long as the previous one. The results from this will be mentioned in a later section.

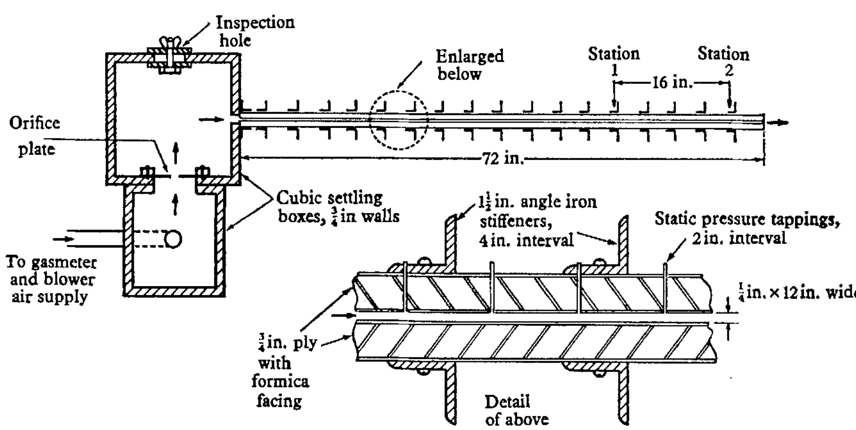

FIGURE 5. Channel layout and details.

The variation of skin-friction coefficient with Reynolds number, based on the average velocity $\overline{U}$ and channel height h, obtained from the 72 in. long channel is shown in figure 6. At Reynolds numbers below about 1300 the results agree closely with the laminar flow relation

$$C_f = 12/Re, \tag{3}$$

while for Reynolds numbers greater than about 2800 the measurements appear to be well represented by

$$C_f = 0{\cdot}0376Re^{-\frac{1}{6}}. \tag{4}$$

These results are qualitatively similar to those obtained in pipes, but the various numerical values are quite different. A survey of the literature indicates that, unlike pipe flow, the fully developed flow in a two-dimensional channel has not been studied at all extensively. Indeed, the recent survey of rectangular duct results made by Hartnett, Koh & McComas (1962) show that there are only two sets of measurements in channels of large aspect ratio which are comparable to the present ones. These are due to Davies & White (1928) and to Washington & Marks (1937). Neither of these results can, however, be regarded as wholly satisfactory. The results of Davies & White are shown by the dashed line in figure 6. From these and an examination of the original paper it was concluded that the measurements of Davies & White must have been made rather close to

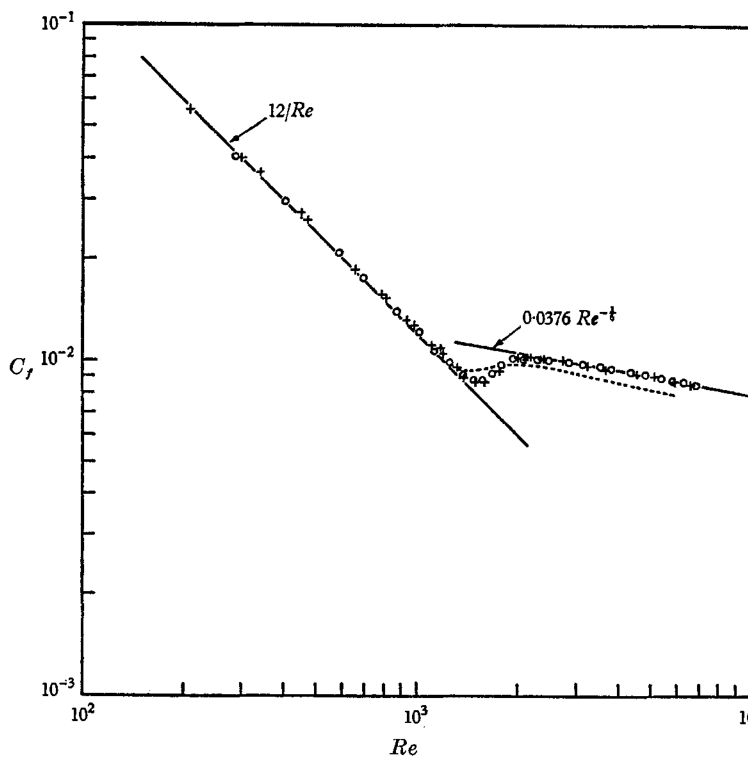

FIGURE 6. Skin friction in channel flow; ○, clear entry; +, $\frac{1}{8}$ in. diameter wire at entry; – – – –, Davies & White (1928).

the entry to the channel so that the flow did not have sufficient length to become fully developed and turbulent. The results of Washington & Marks were also considered but they were rejected as inaccurate due to the fact that the friction measurements were obtained by a rather dubious procedure from the overall pressure drops (i.e. the pressure difference between the settling chambers at entry and exit) rather than the pressure drop over a length of the channel where the flow was fully developed and free from exit or entry effects. There are a few other measurements in parallel channels, such as those of Schlinger & Sage (1953), Laufer (1950), and Robertson (1959), but most of these are restricted to isolated measurements at high Reynolds numbers and none of the authors considers the curve of skin-friction coefficient *versus* Reynolds number around the transition region.

It may also be remarked that the friction laws obtained above are quite different from those which would follow from the use of 'hydraulic mean diameter', a concept which has been found useful in correlating friction factors in pipes of non-circular cross-section. The well known experiments of Nikuradse and Schiller with pipes of rectangular, triangular and trapezoidal cross-sections have shown that the skin friction Reynolds number relationship obtained in circular pipes may also be used for these non-circular pipes provided the Reynolds number is based on the hydraulic mean diameter defined as

$$D_H = \frac{4 \times \text{cross-sectional area}}{\text{wetted perimeter}}. \quad (5)$$

For a two-dimensional channel the hydraulic mean diameter is twice the distance between the parallel plates and simple algebra shows that friction laws for pipes do not reduce to the laws established experimentally. This breakdown of the hydraulic-mean-diameter concept in channels of large aspect ratio has also been commented upon by Leutheusser (1963). In view of this the authors have preferred to define the channel Reynolds number in terms of the channel height h rather than the more conventional length $2h$.

In the present experiments it was found that the (C_f, Re)-curve shown in figure 6 remained unaltered for different entry conditions. Initially the channel was connected directly to the settling box and it was thought that the sharp corners at entry would provide sufficiently disturbed conditions. However, to make quite sure that the flow was fully turbulent, a $\frac{1}{8}$ in. diameter wire was later fitted across the entry, and it was found that this did not alter the results in any way. The approximate value of 1300 mentioned above may therefore be accepted as representing the lower critical Reynolds number for channel flow.

As for the pipe flows described in the last section, mean-velocity profiles were measured at $\frac{1}{8}$ in. from the exit at various Reynolds numbers. A flattened pitot 0·014 in. high was used for this purpose, and the measurements were corrected for pitot-displacement effects by applying a constant correction of 0·15 pitot height as suggested by McQuaid (1966). The measured profiles are shown in figures 7 and 8 in linear and semi-logarithmic co-ordinates respectively. Again, the accuracy of the measurements was checked by confirming that the average velocities computed from the measured velocity profiles were in good agreement with those measured by means of the gasmeter and the orifice.

4. Observation of turbulence in pipes and channels

After the completion of the mean-flow measurements described in the last two sections it was thought desirable to investigate the nature of the intermittent flow in the range of Reynolds numbers over which transition takes place, so as to ascertain the lowest value of the Reynolds number at which intermittency disappears. In pipe flow, similar observations had of course been made previously by Rotta (1956) and others using hot-wire anemometers and by Lindgren (1957) and others using the doubly-refracting property of weak solutions of Bentonite. In the present experiments the signal from a Disa constant-temperature hot-wire anemometer was fed into a Tektronix oscilloscope and the oscilloscope

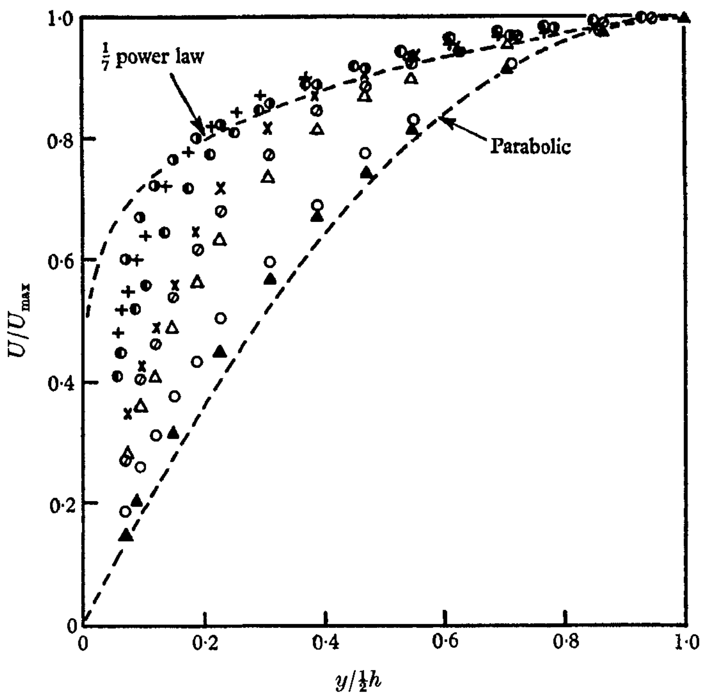

FIGURE 7. Velocity profiles in the $\frac{1}{4}$ in. channel. Values of Re: ▲, 1380; ○, 1580; △, 1725; ⊘, 1835; ×, 2220; ◐, 2760; +, 3650; ◑, 6100.

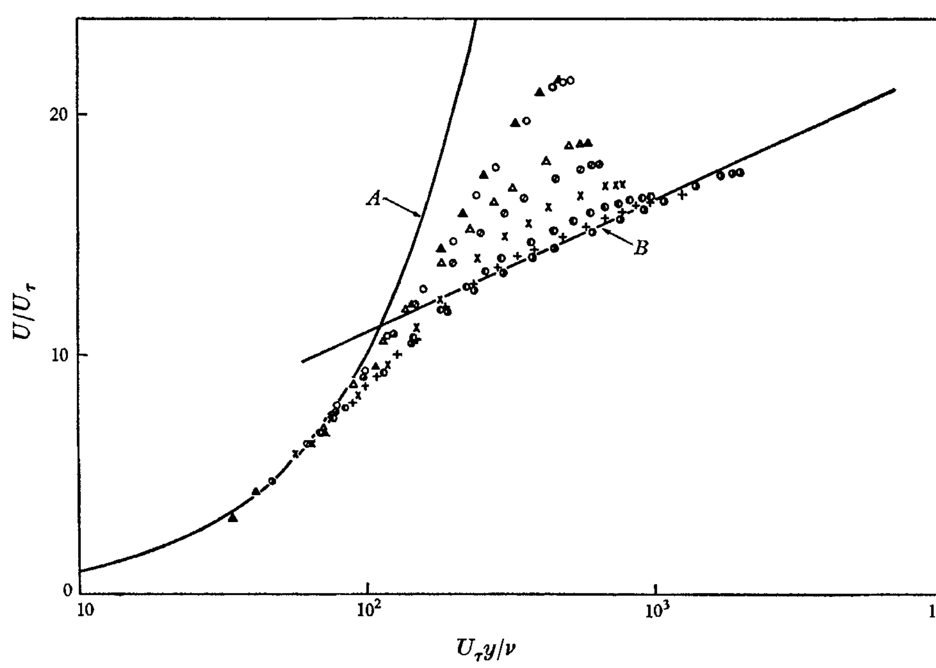

FIGURE 8. Velocity profiles in the $\frac{1}{4}$ in. channel. Values of Re: ▲, 1380; ○, 1580; △, 1725; ⊘, 1835; ×, 2220; ◐, 2760; +, 3650; ◑, 6100. Curve A, $U/U_\tau = U_\tau y/\nu$; curve B, $U/U_\tau = 5{\cdot}5 \log_{10} U_\tau y/\nu + 5{\cdot}45$.

traces were photographed by a Polaroid camera. Typical photographs obtained from the $\frac{1}{4}$ in. and $\frac{1}{2}$ in. diameter pipes are shown in figures 9 to 11. Similar photographs obtained from the $\frac{1}{4}$ in. channel are shown in figure 12.

Figure 9 shows the development of turbulence in pipe flow with increasing Reynolds number. Bursts of turbulence were first observed at a Reynolds number of about 2000. The frequency of occurrence of these bursts increases with increasing Reynolds number, and even though it is difficult to ascertain the precise value of the Reynolds number at which a fully turbulent signal is first obtained, a large number of photographs such as those shown in figure 9 and also similar oscilloscope traces obtained by Rotta (1956) suggest that intermittency disappears when the pipe Reynolds number is in the region of 3000.

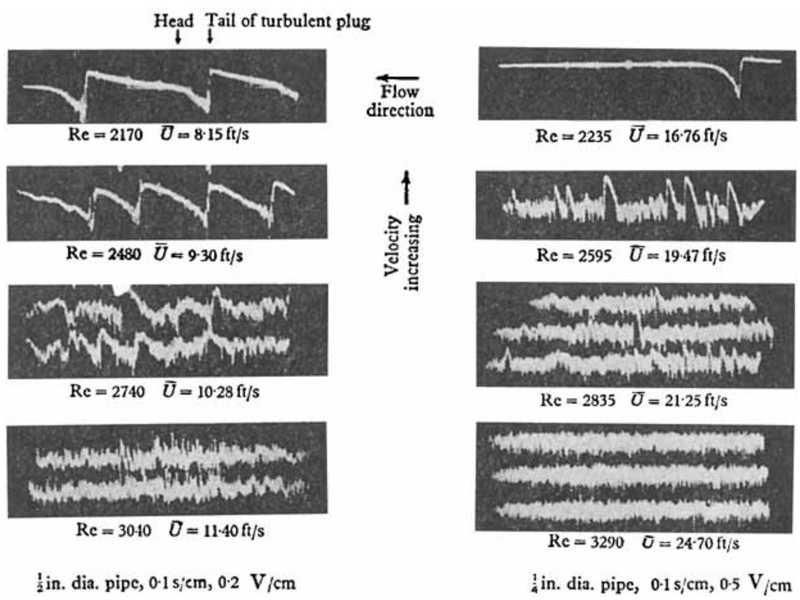

FIGURE 9. Traces from a hot wire at the centreline of the pipes.

The photographs in figure 10 show, qualitatively, the velocity distributions in the laminar and turbulent parts of the fluid. At the centreline of the pipe the velocity in the laminar part of the fluid is higher than that in the turbulent part while very close to the pipe wall the velocity in the turbulent part is higher than in the laminar part. There is also a particular position in the pipe (roughly a distance $0{\cdot}64D/2$ from the centreline) where the laminar and turbulent velocities are equal. It is worth noting that this position coincides roughly with the position where the parabolic laminar profile and a turbulent velocity profile (e.g. the $\frac{1}{7}$ power law) give the same velocity. From this it may be inferred that the velocity distributions in the laminar and turbulent parts of the intermittent flow must be very nearly the same as those occurring in fully developed laminar and fully developed turbulent flow respectively.

The photographs of figure 11, taken with two hot wires, suggest that the turbulent bursts or plugs fill the entire pipe cross-section (except presumably the region immediately adjacent to the wall), so that the intermittency at any particular Reynolds number remains more or less constant across the pipe. Although no quantitative intermittency measurements were made, a large number of photographs similar to those shown in figure 11 support this observation.

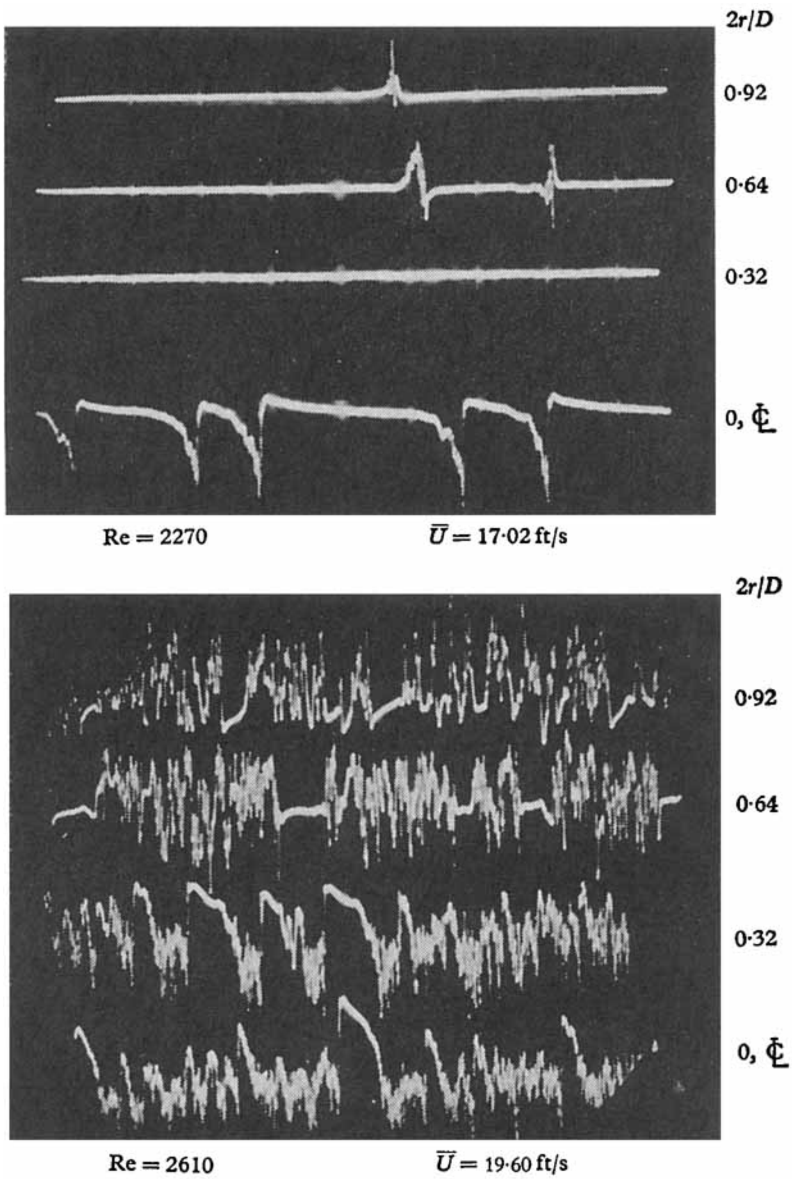

FIGURE 10. Traces from a single hot wire at different locations in the $\frac{1}{4}$ in. diameter pipe. Scales: 0·1 s/cm, 0·5 V/cm.

Turning now to figure 12 obtained from channel flow, it will be seen that a continuously turbulent signal is obtained at $Re = 1975$. Other photographs suggest that the lowest Reynolds number at which a continuously turbulent signal is obtained is in the region of 1800. Turbulent bursts were first observed at a Reynolds number of 1380 so that this figure may be taken as the lower critical Reynolds number for channel flow. Intermittent flow occurs over the Reynolds number range $1380 < Re < 1800$.

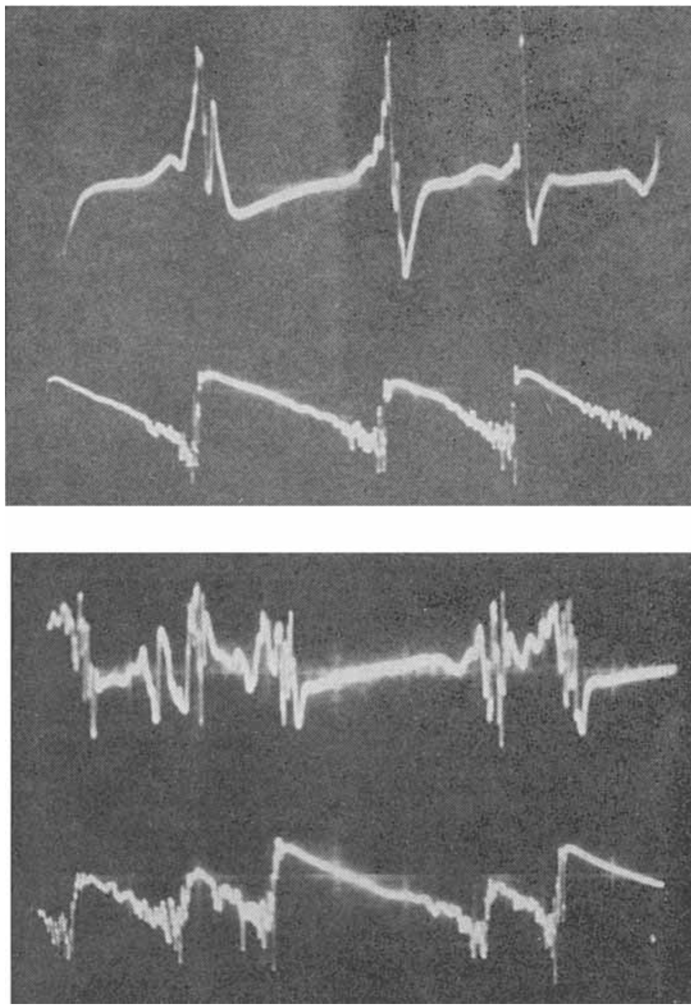

FIGURE 11. Traces from two hot wires in the $\frac{1}{2}$ in. diameter pipe: $Re = 2290$; $\bar{U} = 8{\cdot}60$ ft./s. Scales: 0·1 s/cm, 0·2 V/cm. Upper traces, top for wire at 0·24 in. from the centreline and bottom for wire at the centreline; lower traces, top for wire at 0·18 in. from the centreline and bottom for wire at the centreline.

It will be observed that the distinction between laminar and turbulent signals in channel flow is not as marked as in pipe flow. This is probably because, in the case of pipes, a single turbulent burst fills the entire cross-section of the pipe at any given instant and a hot wire will therefore record signals from one specific burst at any particular time. In the case of channels the situation is more like natural transition in a boundary layer, in so far as turbulent bursts or streaks

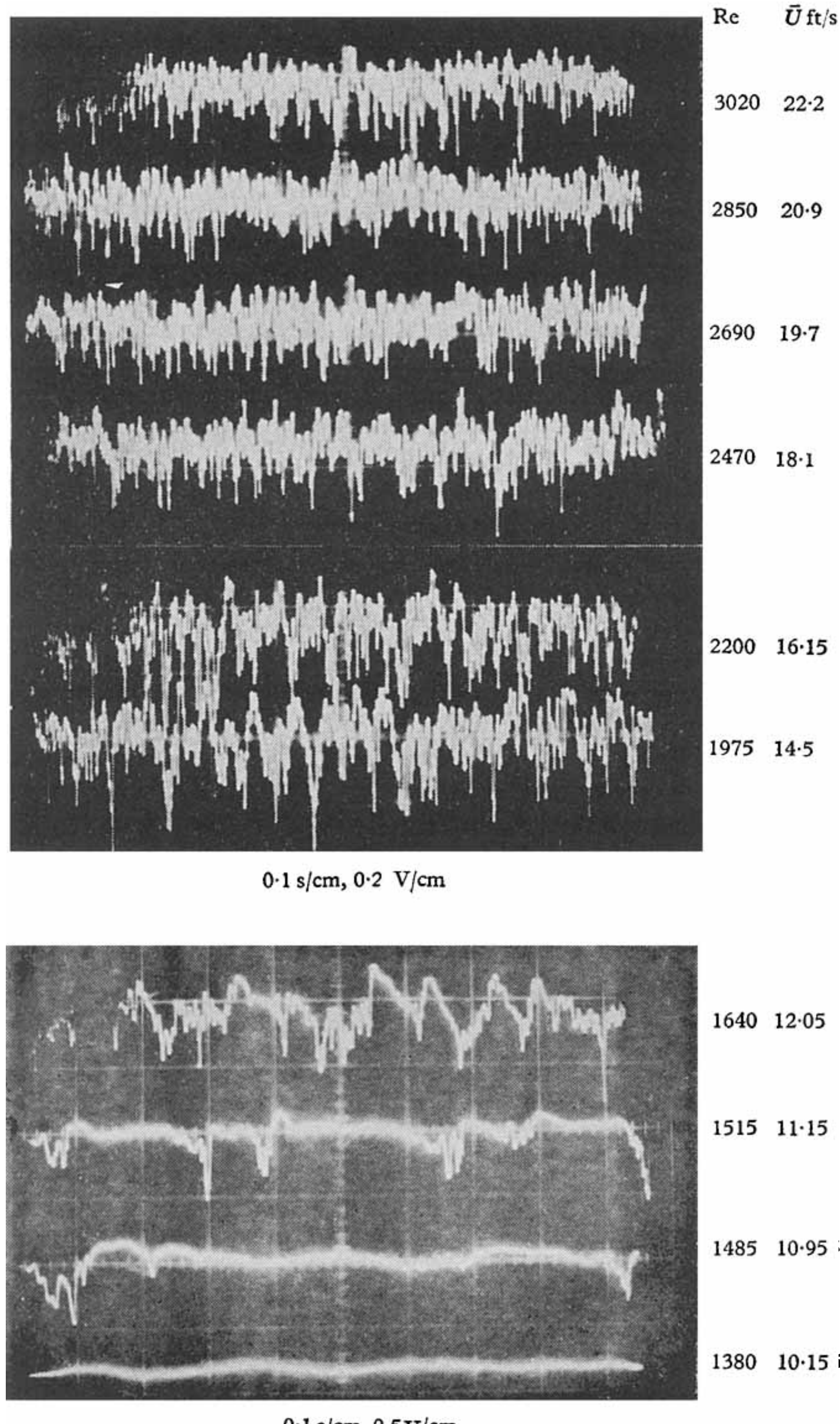

FIGURE 12. Traces from a hot wire at the centreline of the $\frac{1}{4}$ in. high channel.

may be expected to occur randomly, not only in the stream direction but also across the channel, and we may therefore expect several distinct bursts to exist, at any instant, across the width of the channel. Thus the hot wire will be affected not only by bursts whose centreline is oriented along the probe but also by those passing near by. Because of this complication, and also because of the rather limited time scales employed on the oscilloscope, it was thought initially that traces similar to those shown in figure 12 might not have a sufficient resolution, especially at the higher Reynolds numbers, to indicate very small regions of non-turbulent flow. To dispel this doubt, the hot-wire signal was fed into an ultra-violet recorder with a frequency response of up to 5 kc/s and turbulence signals were obtained over much longer time intervals and with various paper speeds. These traces did not, however, reveal any new features and simply confirmed the earlier observation, that fully turbulent signals are obtained at Reynolds numbers greater than about 1800, i.e. intermittency disappears at $Re = 1800$.

5. Criteria for fully developed turbulent flow

In the introduction it was mentioned that, in the past, three criteria have been used to define fully developed turbulent flow. Here we shall examine each of these criteria in turn in the light of the experimental observations described in the preceding sections.

In pipe flow it is normally inferred that transition is completed when the curve of skin-friction coefficient *versus* Reynolds number begins to obey the $\frac{1}{4}$ power friction formula, equation (2), of Blasius. If this criterion is adopted, figure 2 suggests that $Re = 3000$ marks the end of the transition range and that this is the lowest Reynolds number for which fully developed turbulent flow is possible in a pipe. The corresponding criterion applied to the channel results shown in figure 6 leads to the conclusion that the minimum Reynolds number for fully turbulent flow in a channel is in the region 2500 to 3000. In this latter case there is some room for doubt as to the precise value, due partly to the fact that a well established skin-friction relation for turbulent flow is not available and partly to the fact that the experimental data approach a straight line rather more slowly than in the case of pipes.

The second criterion used to characterize fully developed turbulent flows is that the velocity distribution in the wall region should follow the well known semi-logarithmic law,

$$U/U_\tau = A \log_{10} U_\tau y/\nu + B, \tag{6}$$

where A and B are universal constants which take values in the region of 5·5 and 5·45 respectively (Patel 1965). If the existence of such a universal logarithmic region is taken as a necessary condition for fully turbulent flow, then figures 4 (*a*) and 4 (*b*) suggest that fully turbulent flow is not established even at Reynolds numbers as high as 9200. Figure 4 shows that, above a Reynolds number of about 3000, the velocity profiles follow the semi-logarithmic trend indicated by equation (6) but the value of the constant B is different from, and appreciably larger than, the universal value quoted above. In fact B decreases with increasing Reynolds number as shown in figure 13 and tends to the value 5·45 obtained in pipes at high Reynolds numbers and in flat-plate turbulent boundary layers (Patel 1965) for Reynolds numbers approaching 20 000. We may digress here to note another interesting feature of pipe flow. Figure 14 shows the measured mean-velocity profiles for pipe Reynolds numbers between 3000 and 9200 compared with the $\frac{1}{7}$ power law. It will be recalled from figure 2 that within this

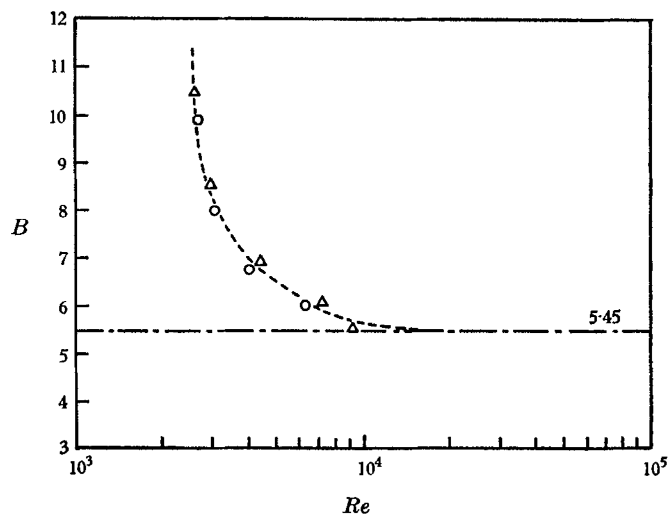

FIGURE 13. Variation of B with Re: ○, $\frac{1}{4}$ in. diameter pipe; △, $\frac{1}{2}$ in. diameter pipe.

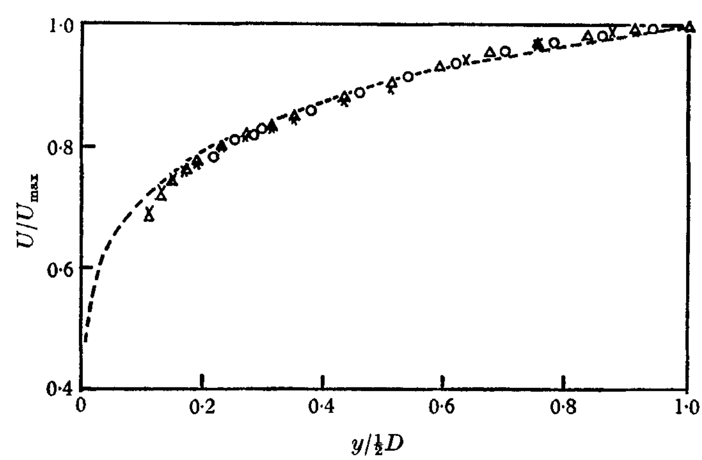

FIGURE 14. Comparison with the $\frac{1}{7}$ power law: ×, Re = 9200, $\frac{1}{2}$ in. pipe; △, Re = 7260, $\frac{1}{2}$ in. pipe; ○, Re = 6300, $\frac{1}{4}$ in. pipe ; – – – –, $\frac{1}{7}$ power law.

range of Reynolds number the skin-friction measurements are in excellent agreement with the Blasius $\frac{1}{4}$ power friction law of equation (2). Now, reference to any standard text-book (e.g. Schlichting 1962) shows that the $\frac{1}{4}$ power law for skin friction is compatible with a $\frac{1}{7}$ power law velocity distribution, and indeed, by an overlap analysis, it is possible to derive the one law from the other, apparently quite rigorously. It is therefore surprising to find that the measured profiles are not in agreement with the $\frac{1}{7}$ power profile even though the skin-friction measurements follow the $\frac{1}{4}$ power law. In fact it is seen from figure 14 that the $\frac{1}{7}$ power law is not a good approximation even for Reynolds numbers up to 9200, though the region of agreement increases from the pipe centre outwards as the Reynolds number is increased, the region of agreement being $0{\cdot}2 < y/\frac{1}{2}D < 1{\cdot}0$ for $Re = 9200$. Figure 15 shows the variation of the measured values of $\overline{U}/U_{\max}$, $\overline{U}$ being the average velocity and $U_{\max}$ the maximum velocity occurring at the centre of the pipe. For pipe flow this ratio takes the value 0·5 for the laminar parabolic profile and 0·817 for the turbulent $\frac{1}{7}$ power law profile. From the figure it is clear that the velocity profiles closely approach the $\frac{1}{7}$ power law only for Reynolds numbers greater than 10 000.

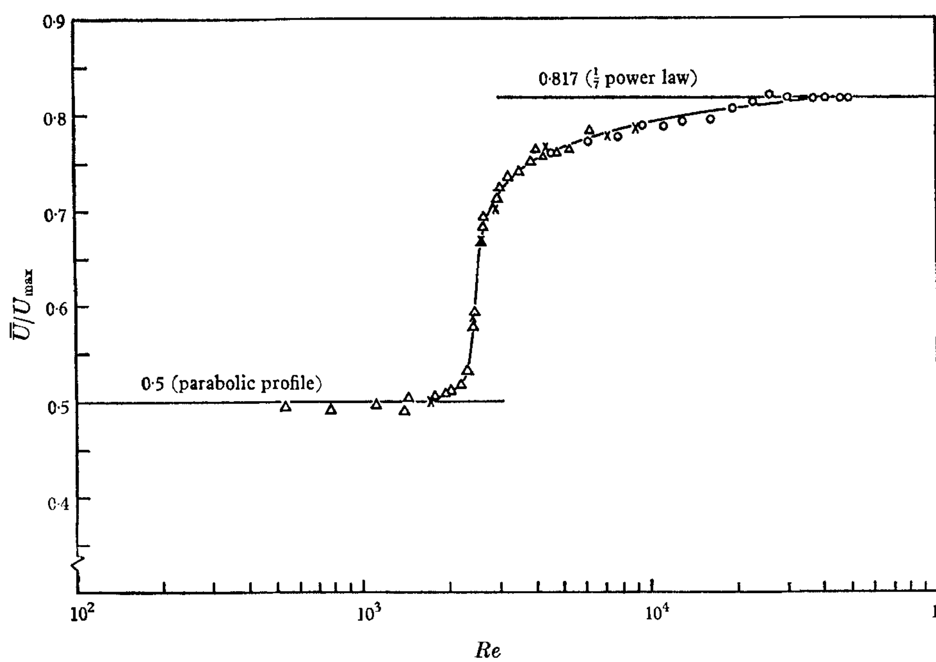

FIGURE 15. Ratio of mean to centreline velocity in pipe flow: △, $\frac{1}{4}$ in. pipe; ×, $\frac{1}{2}$ in. pipe; ○, 2 in. pipe.

Applying the criterion of a universal inner law to the channel results shown in figure 8, we find that the flow becomes fully turbulent at Reynolds numbers greater than about 3000.

Considering next the third criterion, that of the disappearance of intermittency, we note from the results presented in §4 that in pipe flow a fully turbulent signal is first observed at $Re = 3000$ while in channel flow the corresponding Reynolds number is in the region of 1800.

The above results are shown summarized in table 1. From these results it is clear that the three criteria do not lead to a unique minimum value of Reynolds number above which fully developed turbulent flow can be said to exist for either pipe flow or channel flow. In each case two of the criteria lead to the same result, but the criterion that must be rejected is different in the two cases. There is therefore some doubt concerning the definition of fully developed turbulent flow, and the precise value of the lowest Reynolds number at which it is first observed.

Criterion	Reynolds number: Pipe flow	Reynolds number: Channel flow
C_f–Re skin-friction relation	3000	2500–3000
Log-law with universal constants	>10000	~ 3000
Disappearance of intermittency	3000	~ 1800

TABLE 1

The observation that the curve of skin friction *versus* Reynolds-number and the disappearance of intermittency both suggest the value of 3000 as the lowest Reynolds number for fully developed turbulent flow in a pipe is in agreement with the earlier results of Binnie & Fowler (1947), Rotta (1956), Lindgren (1957) and many others. The failure of the flow in the wall region to conform to the accepted law of the wall even at Reynolds numbers considerably above 3000 has not, so far as the authors are aware, been previously noted. Indeed, the measurements of Nikuradse (quoted by Schlichting 1962) which include only one velocity profile in the Reynolds number range of interest here, namely $Re = 4000$, appear to demonstrate agreement with the law of the wall at Reynolds numbers as low as 4000.

In the case of channel flow no previous measurements or observations are available which can be compared directly with the present results, although the investigations of Badri Narayanan (1966, 1968) have some relevance. Badri Narayanan made mean-velocity and turbulence measurements in a $\frac{1}{2}$ in. deep channel, downstream of a diffuser in which the width increased from 3 to 9 in. over a distance of 48 in. With this increase in width the Reynolds number was reduced by a factor of three. Upstream of the diffuser the flow was fully developed and turbulent and measurements were made at four different downstream Reynolds numbers, namely 1250, 1730, 1960 and 2500. Measurements of turbulence intensity and shear stress showed that the flow downstream of the diffuser was reverting, asymptotically, to the laminar state in all four cases. The decay of turbulence with distance from the end of the diffuser was found to be slowest in the case of the flow with the highest Reynolds number, as might be expected, and extrapolation of the rates of decay to zero suggests that at a Reynolds number of about 2900 the flow should be in energy equilibrium and the turbulent flow self-maintaining (see figure 16 where Badri Narayanan's results have been reproduced).

At first sight, this would appear to confirm the present results, in suggesting that self-maintaining turbulent channel flow cannot exist below a Reynolds number of about 2900, a value which is in excellent agreement with that obtained from both the C_f–Re criterion and that based on the law of the wall. However, upon closer examination, the present observations must be interpreted as showing quite clearly that continuously turbulent channel flow, which may be assumed to be in the fully developed condition (i.e. independent of downstream distance), occurs at considerably lower Reynolds numbers.

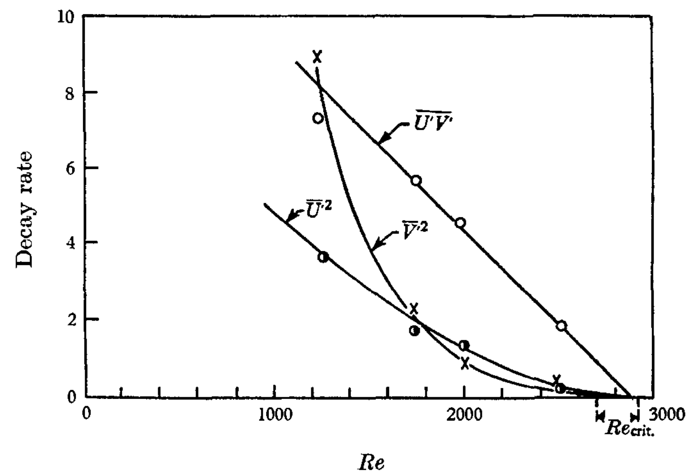

FIGURE 16. Determination of the critical Reynolds number in channel flow (Badri Narayanan 1966).

The question might be asked, whether the present observations in the $\frac{1}{4}$ in. channel were in fact made in fully developed conditions, and whether the turbulence was truly self-maintaining at Reynolds numbers as low as 1800. To answer this question quite unequivocally the channel length was doubled and the hot-wire observations close to exit repeated. No differences could be discerned in the oscillograph traces or in measurements of turbulent intensity and the C_f–Re relation.

Experiments similar to those of Badri Narayanan have been performed in pipes by Sibulkin (1962) and Laufer (1962), but only Sibulkin's results are relevant here. His measurements show that, at the one Reynolds number (2400) for which tests were conducted in the transition régime, the mean turbulence intensity first decayed downstream of the diffuser before increasing to a final equilibrium value. This may go some distance towards explaining the discrepancy between the present channel results and those of Badri Narayanan.

6. Discussion

We may sum up the results of the previous sections as follows. It appears that, for fully developed pipe flow with disturbed entry conditions, the upper limit of what is normally regarded as the transition régime is characterized by the disappearance of intermittency and by a rapid adjustment to the accepted Blasius skin-friction law for fully turbulent flow. At the same time, however, for Reynolds numbers considerably greater than this value, neither the $\frac{1}{7}$ power law for velocity distribution (which is normally accepted as being associated with the $\frac{1}{4}$ power skin-friction law) nor the accepted logarithmic law of the wall (which is associated with the logarithmic skin-friction law) is obeyed.

On the other hand, for fully developed channel flow with disturbed entry conditions, the disappearance of intermittency occurs at a very much lower Reynolds number (1800) than does the adjustment to a regular skin-friction law (at $Re = 2800$, say), and the relative gradualness of the adjustment may well be a reflexion of this fact. It therefore appears that self-maintaining turbulent flow in energy equilibrium may occur in a range of Reynolds numbers where no recognizable law of the wall exists and where no regular skin-friction law applies. In channel flow, however, unlike pipe flow, once a logarithmic velocity distribution can be distinguished, at Reynolds numbers greater than about 2800, the constants associated with it are not noticeably different from the accepted values appropriate at much higher Reynolds numbers.

At least some of the differences between pipe and channel results may be explained by the differences between axisymmetric and two-dimensional flow. Millikan's (1938) derivation of the logarithmic form of the law of the wall depends upon two assumptions, that the velocity distribution sufficiently close to the wall, expressed in suitable non-dimensional terms, is independent of pipe radius, and that the distribution of velocity defect in the turbulent core, again expressed in suitable non-dimensional terms, is independent of viscosity. The further assumption, that there is an overlap region in which both the wall law and the velocity defect law apply, leads directly to logarithmic expressions with universal constants for the law of the wall and the velocity-defect law. Now, the assumption that the pipe radius *per se* is unimportant in determining the velocity distribution close to the wall would appear necessarily to restrict the application of the law of the wall to only a small fraction of the pipe radius from the wall, and, if an overlap region is to exist, the region of flow which is substantially independent of viscosity must extend to within this small distance from the wall. It seems quite plausible that this condition should obtain only at Reynolds numbers considerably higher than 3000; at lower Reynolds numbers a logarithmic region may exist but the constants will not have the universal values.

If similar considerations are applied to the case of channel flow, there would appear to be no reason *a priori* for restricting the application of the universal law of the wall to small distances from the wall, and the existence of any appreciable central region where the flow is unaffected by viscosity might be expected to lead to an overlap region in which the velocity distribution follows the universal form. This provides a tentative explanation for the observed conformity with the universal law of the wall in the lower range of Reynolds numbers but leaves other questions unanswered. For example, it will be recalled that the $\frac{1}{5}$ power law for skin friction in the flat-plate boundary layer, which is generally accepted as being reliable in the lower range of Reynolds numbers, was originally derived by the direct application of pipe-flow data, including the $\frac{1}{4}$ power law for skin friction. The use of channel data, which at first sight would appear to be more relevant, would lead to a different result since the friction coefficient follows (as nearly as can be ascertained) a $\frac{1}{6}$ power law.

7. Conclusions

The following are the significant results of the present investigation.

Pipe flow

(i) It is confirmed that, with disturbed entry conditions, the transition régime is initiated at a Reynolds number of approximately 2000 and is terminated, fairly abruptly, at a Reynolds number in the region of 3000.

(ii) Up to this Reynolds number the flow is only intermittently turbulent.

(iii) Above this Reynolds number the flow is continuously turbulent and the $\frac{1}{4}$ power law for skin friction is obeyed.

(iv) In the lower range of Reynolds numbers above 3000 (up to 10 000 say) the velocity distribution across the pipe is poorly described by the $\frac{1}{7}$ power law and there is thus no necessary connexion between this power law and the $\frac{1}{4}$ power law for skin friction.

(v) Over a similar range of Reynolds numbers the velocity distribution is substantially logarithmic outside the sublayer and blending region but the additive constant is different from that in the universal law of the wall and approaches it closely only at Reynolds numbers above about 10 000. This finding does not accord with the results of Nikuradse.

Channel flow

(i) From a plot of skin-friction coefficient against Reynolds number, the transition régime in channel flow with disturbed entry conditions appears to extend from a Reynolds number of approximately 1350 to a Reynolds number in the range 2500–3000, the upper limit being much less clearly defined than for pipe flow.

(ii) The flow is continuously turbulent for Reynolds numbers greater than about 1800, i.e. well within the transition régime as determined from the C_f–Re plot.

(iii) For Reynolds numbers greater than about 2800 the results for skin friction conform to the power-law relation $C_f = 0{\cdot}0376/Re^{\frac{1}{6}}$.

(iv) At these Reynolds numbers the velocity distribution outside the sublayer and blending region is substantially logarithmic and the constants take the values associated with the universal law of the wall.

A general outcome of the investigation is the difficulty of defining a unique criterion by which to specify fully developed turbulent flow.

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